More on Boltzmannia

→ Summary of Basic Boltzmannia.

– Basic Eqn.

$$C(f) = \int d\underline{p}_2 \, \frac{\partial \underline{V}_{1,2}}{\partial \underline{r}_1} \cdot \frac{\partial}{\partial \underline{p}_1} \left[f(1,t)\, f(2,t) \right]$$

$\uparrow$ Collision operator

$$= \int d\underline{p}_1 \int d\underline{p}' \int d\underline{p}_1' \; w(\underline{p}', \underline{p}_1'; \underline{p}_1, \underline{p}) \left(f(\underline{p}')\, f(\underline{p}_1') - f(\underline{p}_1)\, f(\underline{p}) \right)$$

with

$$w \equiv w(\underline{p}', \underline{p}_1'; \underline{p}_1, \underline{p}) = w^T \quad \text{(time reversible)}$$

as transition probability

⇒ B.E.

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f = C(f)$$

based on
- detailed balance $w^T = w$
- molecular chaos

Now, $\sigma = -\int d\underline{x} \int d\underline{p}\; f \ln f$ → entropy.

and : $$\frac{dS}{dt} = \frac{1}{2} \int d^4\underline{p} \int d^3\underline{x} \; w \, f f_1 \left[x \ln x - x + 1 \right]$$

$$x = f' f_1' / f f_1$$

$\frac{dS}{dt} \geq 0$ $\Rightarrow$ H thm.

Max entropy state, $x=1$ $\Rightarrow dS/dt = 0$.

$\Rightarrow$ Some observations:

(a)

→ never actually used concept of equilibrium or equilibrium distribution function in building Boltzmannia, though did observe $C(f_{eq}) = 0$.

→ might ask: "If no a-priori knowledge of equilibrium distribution, could one derive it?"

Now:

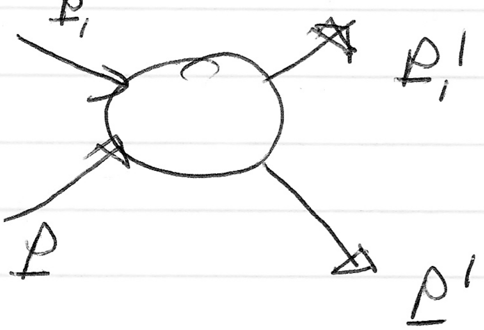
$$\frac{dS}{dt} = 0 \quad \text{for} \quad x=1 \quad \left\{ \begin{array}{l} \text{point of} \\ \text{maximal} \\ \text{entropy} \end{array} \right.$$

$x=1 \Rightarrow$ $$f'f_1' = ff_1$$

$$\ln f' + \ln f_1' = \ln f + \ln f_1$$

as labels in collision are arbitrary,

i.e.



have, at equilibrium:

$$\ln f + \ln f_1 = \text{const}$$

{ for equilibrium ($ds/dt = 0$)

⇒ { sum of logs conserved in collision.

Now, what is constant in collision:

- energy (kinetic for pt. particle)
- momentum
- number

⇒ $\ln f$ can be expressed as a linear combination (with constant coeffs) of quantities conserved in a collision.

so

$$\ln f = \underline{a} + \underset{\substack{\downarrow \\ \text{momentum}}}{\underline{b}\cdot\underline{p}} + c\underset{\substack{\downarrow \\ KE}}{\frac{\underline{p}^2}{2m}}$$

Note: - $c < 0$ for normalizability of f

- conservation requires form of $\ln f$

- angular momenta not independent, as collision event occuring at 1 position.

$$\Rightarrow f = c' \exp\left[\frac{-\underline{p}^2}{2mT} - \frac{\underline{p} \cdot \underline{V}}{T} \right]$$

→ recovers Maxwell-Boltzmann distribution

→ n, T, V all can be functions of x for $l_{mfp} < L$

→ have derived form of eqbrm distribution from condition $dS/dt = 0$

ⓑ

→ Boltzmann H-Thm exploited concept of entropy. Where from?

Fundamentally, Entropy $\Leftrightarrow$ $\ln$ (Phase Volume)

⊗ $$\boxed{\sigma = \ln \Delta\Gamma}$$ → Fundamental definition

$$= \ln \frac{\Delta p \Delta q}{(2\pi\hbar)^s}$$

↳ ensures dimless.

Now, $f(\bar{E})\,\Delta\Gamma = 1$

f: weight factor; $\bar{E}$: mean energy ⇒ assumes tightly localized f [illegible]

and linearity of log:

$$\ln f(E) = \alpha + \beta E$$ (uses struct. of Eqbrm)

so $$\ln f(\bar{E}) = \alpha + \beta \bar{E}$$ → can obtain from Liouville Eqn, with sub-systems.

⇒
$$\ln f(\bar{E}) = \alpha + \beta \bar{E} = \langle \ln f(E) \rangle$$

thus

$$\sigma = \ln \Delta\Gamma = -\ln f(\bar{E}) = \ln \langle f(E) \rangle$$

⇒

$$\sigma = -\langle \ln f(E) \rangle = -\int f \ln f \, d\Gamma.$$

⇒ recovers entropy, used in H-Thm.

ⓒ How Reconcile?

- <u>reversible</u> laws of Hamiltonian mechanics, which govern gas
- $dS/dt \geq 0$.

related:

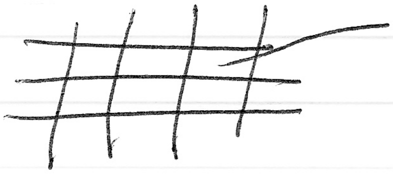
- whatever happened to Poincare recurrence?

Key Point: Boltzmann introduces:

→ statistical description
i.e. $f(\underline{x}, \underline{p}, t)$ as probability

→ coarse graining (recall Lyapunov exponents!)

<u>i.e.</u> partition



$\Delta p, \Delta x.$

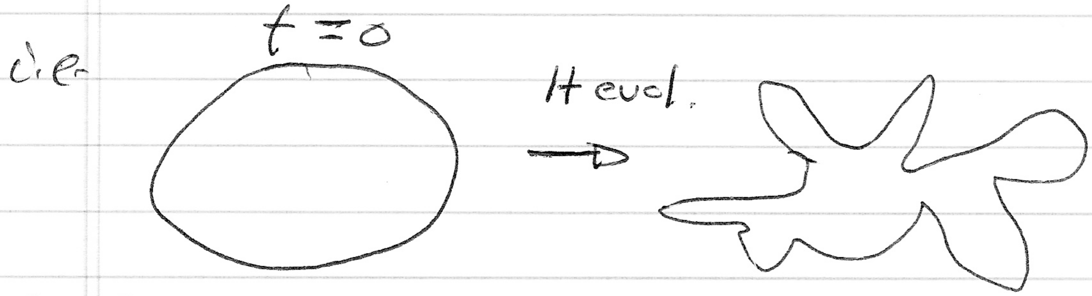
↓

i.e. resolution scale

(recall S integrated)

Why significant?

⇒ partition / coarse graining kills small details on phase volume evolution



zapped
(i.e. phase space
reconnection) by
coarse graining!

⇒

prediction of very low probability events is impossible, inaccurate.
and

⇒ recurrence is very low probability.

ⓓ Why is statistical description valid?

→ chaos (even for $N=2$, not only $N=N_A$!)

i.e. $\tau_{relax} \gg \underset{\text{mixing time}}{\underset{\downarrow}{\tau_{Lyap}}}$

i.e. calculated D_M → it works.

→ what are the key assumptions:

→ reversible, conservative collisions

→ $f(1,2) = f(1)\, f(2)$
(Molecular chaos)

[chaos → correlation
mixed

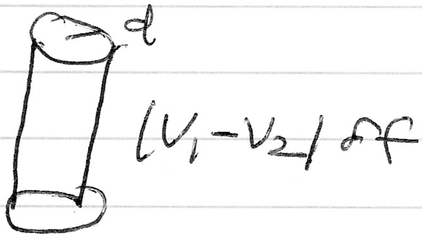
Per Boltzmann: <u>Stosszahlansatz</u>

i.e.

total # $(\underline{v}_1, \underline{v})$ collisions taking place in δt

= Volume of $(\underline{v}_1, \underline{v})$ collision cylinder
$\circledast$ # particles with v_1, per volume.

collision cylinder:



in collision cylinder $= v_{rel}\, \sigma\, f(\underline{v}_1)\, d^3v_1$

N.B.:

- dilute → non-overlapping cylinders
- collisions as 'point events'

  $d < \bar{r} < l_{mfp}$ ordering!
- $f(1,2)$ factorization

all buried here.

Also

→ $w\, d^3v'\, d^3v_1' = v_{rel}\, d\sigma$

relates transition to familiar items like cross-section.

→ for l_{mfp}, $l_{mfp} = 1/n\sigma$

$\sigma\, l_{mfp}$ = volume of collision cyl., for 1 collision.

$l_{mfp}\,\sigma\, n = 1$

i.e $n\,\sigma\, l_{mfp}$ = # coll = 1

$$\boxed{l_{mfp} = 1/n\sigma}$$

$\nu_{coll} = v_{Th}/l_{mfp}$ → defines collision frequency.

n.b. crudely: $\sigma \sim d^2$.

→ Fluid Equations

- what to do with B.E. ⇒

$$\begin{cases} \text{fluid eqns.} \\ \text{transport coefficients} \rightarrow \text{flux – force proportionality} \end{cases}$$

- replace B.E. by set of equations which evolve thermodynamic parameters.

⇒ hydrodynamic equations for $n(\underline{x},t)$, $\underline{V}(\underline{x},t)$, $P(\underline{x},t)$ etc.

- local, Eulerian description (lab frame)
  ↳ held together by collisions, vs. dispersion
- describes 'blobs' of gas, held together by collisions

$$\left.\begin{aligned} L &> l_{mfp} \\ \omega &< \nu_{coll} \end{aligned}\right\} \text{scales of relevance}$$

- parametrizes dynamics in terms structure of distribution

$$f = \frac{n(\underline{x})}{(2\pi)^{3/2}\, v_{th}(\underline{x})^3} \exp\left[-\left(\underline{v}-\underline{V}(\underline{x},t)\right)^2 / v_{th}^2(\underline{x},t)\right]$$

- works for slight deviation from equilibrium i.e.

$$\begin{tikzcd}
f = f_0 \arrow[d] & + \delta f \arrow[r] & \delta f = -\frac{\underline{v}\cdot\nabla f_0}{\nu} \arrow[d, Rightarrow] \\
\text{local Maxwellian} \arrow[d, Rightarrow] & & \text{viscous dissipative eqns.} \\
\text{ideal equations} & &
\end{tikzcd}$$

— sacrifices info re: higher moments (usual energy truncation).

now, recall:

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f = C(f)$$

demand: $\int d^3\underline{v}\, C(f) = 0$ (# conservation)

$\int d^3\underline{v}\, m\underline{v}\, C(f) = 0$ (momentum conservation)

$\int d^3\underline{v}\, \varepsilon\, C(f) = 0$ (energy conservation).

then define:

$n = \int d^3v\, f$ → density

$\underline{V} = 1/n \int d^3v\, \underline{v}\, f$ → velocity / momentum

$$\bar{\epsilon} = \frac{1}{n}\int d^3\underline{v}\; \epsilon f$$ → energy density

and $$\partial_t f + \partial_{x_i}(v_i f) = C(f)$$

taking moments;

$$\frac{\partial n}{\partial t} + \underbrace{\nabla\cdot(n\underline{v})}_{\text{mass flux}} \underbrace{= 0}_{\#\text{ cons.}}$$

$$\frac{\partial}{\partial t}(m v_\alpha) + \underbrace{\frac{\partial}{\partial x_\beta}\Pi_{\alpha,\beta}}_{\text{momentum flux}} \underbrace{= 0}_{\text{mom. cons.}}$$ (momentum balance)

$$\Pi_{\alpha,\beta} = \int d^3v\; m v_\alpha v_\beta f$$

$$\frac{\partial(n\bar{\epsilon})}{\partial t} + \underbrace{\underline{\nabla}\cdot\underline{q}}_{\text{energy flux}} \underbrace{= 0}_{\text{energy cons.}}$$ (energy balance)

$$\underline{q} = \int d^3\underline{v}\; \underline{v}\,\epsilon f$$

note form:

$$\frac{\partial}{\partial t}(\text{stuff}) + \underline{\nabla}\cdot(\text{Flux of stuff}) = 0$$

→ essence is calculation of fluxes.

Furthur simplify by:

$$\underbrace{\underline{v} = \underline{V}(\underline{x},t)}_{\text{mean/bulk flow}} + \underbrace{\underline{v}'}_{\text{thermal fluctuation about mean}}$$

mean/bulk flow (~ linked to body forces)

thermal fluctuation about mean $(\sim \sqrt{T/m})$

realistically: $|\underline{V}| \ll |\underline{v}'|$

but $\underline{v}'$'s cancel

so

$$\Pi_{\alpha,\beta} = \int d^3v\; m\,(V_\alpha(\underline{x},t) + v'_\alpha)\,(V_\beta(\underline{x},t) + v'_\beta)\, f$$

$$f = f_0 + \cancel{\delta f}$$ ↗ here

f_0 → loc. Maxwellian, as above.

so

$$\Pi_{\alpha,\beta} = mn\left(V_\alpha(\underline{x},t)\,V_\beta(\underline{x},t) + \langle v'_\alpha v'_\beta \rangle\right)$$

now, $$f = \frac{n(\underline{x})}{(2\pi)^{3/2} v_{Th}^{3}(x)} \exp\left[-\frac{(\underline{v}-\underline{V}(\underline{x},t))^2}{v_{Th}(x)^2}\right]$$

$\langle v'_\alpha v'_\beta \rangle = 1/3\, v'^2 \delta_{\alpha,\beta}$ (~isotropic f_0, of $\underline{v}$)

$\langle v'^2 \rangle = 3T/m$

(collisions isotropize pressure)

so, can define:

$$\underline{\underline{P}} = mn\langle v'_\alpha v'_\beta \rangle$$

$$= \frac{1}{3} mn \langle v'^2 \rangle \delta_{\alpha,\beta}$$

$$p = nT$$

so

$$\frac{\partial}{\partial t}(n\underline{V}) + \underline{\nabla}\cdot\left(n\underline{V}\,\underline{V} + \underline{\underline{I}}\,p\right) = 0$$

but: $$\frac{\partial n}{\partial t} + \nabla\cdot(n\underline{V}) = 0$$

subtracting →

$$n\left(\frac{\partial}{\partial t}\underline{V} + \underline{V}\cdot\underline{\nabla}\underline{V}\right) = -\underline{\nabla}p$$

"Euler Eqn."

(+visc → Navier-Stokes).

$\underline{\nabla}\cdot\underline{V} = 0$ defines p (incompressible fluid).

Similarly,

$$\epsilon = \tfrac{1}{2}mv^2 + \epsilon'$$

$$= \frac{1}{2}m\left(\underline{V}(\underline{x},t) + \underline{v}'\right)^2 + \epsilon'$$

so

$$\underline{Z} = \int \epsilon\,\underline{v}\,f\,d^3\underline{v}$$

$$\cong \int \epsilon\,v\,f_0\,d^3\underline{v}$$

$$\underline{Z} = \int d^3\underline{v}\,\left(\underline{V}(\underline{x},t) + \underline{v}'\right)\left(\epsilon' + \tfrac{1}{2}m\left(\underline{V}(\underline{x},t) + \underline{v}'\right)^2\right)f$$

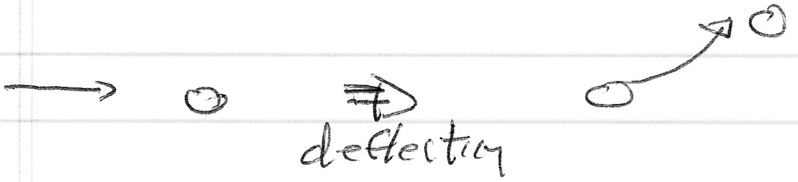
$$= \underline{V}(\underline{x},t)\left(\tfrac{1}{2}nmV^2 + \underbrace{p + n\bar{\epsilon}'}_{\hookrightarrow\, w \,\to\, \text{internal energy}}\right)$$

so

$$\frac{\partial}{\partial t}(n\epsilon) + \underline{\nabla}\cdot\left[\underline{V}(\underline{x},t)\left(\frac{1}{2}mnV^2 + p + n\bar{\epsilon}'\right)\right] = 0$$

↓ can simplify as for momentum.

→ angular momentum <u>not</u> conserved by $C(f)$ |



→ most truncations stop at 3rd moment

Now, for transport:

→ what if f_0 is gradient
⇒ drives δf

i.e. Chapman-Enskog expansion
⇒ $\delta f = -\frac{1}{\nu} \underline{v} \cdot \underline{\nabla} f_0$

→ then $f_0 + \underline{\underline{\delta f}}$ set fluxes, so

$$\text{Flux} = \underbrace{\text{ideal part}}_{f_0} + \underbrace{\text{collisional part}}_{\delta f \,\Rightarrow\, \text{transport}}$$

Key Point: How relate fluxes to gradients in thermodynamic quantities?